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On the asymptotic stability of equilibrium ☆

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Abstract

The classical criterion of asymptotic stability of the zero solution of equations $x' = f(t,x)$ is that there exists a positive definite function V which has infinitesimal upper bound such that $\frac{dV}{dt}$ is negative definite. In this paper we prove that if $\frac{d^{m+1}V}{dt^{m+1}}$ is bounded then the condition that $\frac{dV}{dt}$ is negative definite can be weakened and replaced by that $\frac{dV}{dt} \leqslant 0$ and $-(|\frac{dV}{dt}| + |\frac{d^2V}{dt^2}| + \cdots + |\frac{d^mV}{dt^m}| + |\frac{d^{m+p}V}{dt^{m+p}}|)$ is negative definite.

Keywords: Asymptotic stability; Instability; Lyapunov's direct method

1. Introduction

Consider a system of ordinary differential equations

$$\frac{dx}{dt} = f(t,x), \tag{1}$$

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where $x \in B_H$, $B_H = \{x \in R^n\colon \|x\| \leqslant H\}$; the function $f : [0, +\infty) \times B_H \to R$ is smooth enough to ensure existence and uniqueness of the solution of the initial value problem associated with (1); $f(t, 0) \equiv 0$, $t \geqslant 0$.

The classical criterion of asymptotic stability of the zero solution of equations $x' = f(t, x)$, which was obtained by Lyapunov [1], is that there exists a positive definite function V which has infinitesimal upper bound such that $\frac{dV}{dt}$ is negative definite. In applications one often constructs a positive definite V which derivative is not negative definite, but is less than or equal to zero. Exactly for such cases Barbashin and Krasovski created an asymptotic stability criterion [1], but it has a drawback, namely that the criterion is effective only when Eq. (1) is an autonomous system or a periodic system. In this paper, we prove that for nonautonomous system (1), if there exists a positive definite function V which has infinitesimal upper bound such that $\frac{dV}{dt} \leqslant 0$, $\frac{d^{m+1}V}{dt^{m+1}}$ is bounded, and $-\big(\big|\frac{dV}{dt}\big| + \big|\frac{d^2V}{dt^2}\big| + \cdots + \big|\frac{d^mV}{dt^m}\big| + \big|\frac{d^{m+p}V}{dt^{m+p}}\big|\big)$ is negative definite then the zero solution of Eq. (1) is asymptotically stable. We also give a corresponding instability criterion and present some examples showing the applications of these new criteria.

2. Some lemmas

First, we introduce some lemmas, they are all simple propositions in analysis, but they play a key role in proving the theorems in this paper.

Lemma 1. *Consider a C^m function $f : [0, +\infty) \to R$, if the limit $\lim_{x\to+\infty} f(x)$ exists, then for every sequence $\{y_k\}$ with $y_k \to +\infty$ $(k \to \infty)$ there exists a sequence $\{x_k\}$ with $x_k - y_k \to 0$ $(k \to \infty)$ such that*

$$\lim_{k\to\infty} f^{(r)}(x_k) = 0, \quad 1 \leqslant r \leqslant m.$$

Proof. This proof is done by contradiction. Suppose that this conclusion does not hold, then there exist $a > 0$ and $\delta > 0$ such that for arbitrary large $X > 0$ there exists $y_N > X$ ($y_N \in \{y_k\}$) such that for every $x \in [y_N, y_N + \delta]$, $|f^{(r)}(x)| > a$ holds. By the mean value theorem, we have

$$\big|f^{(r-1)}(x)\big| > \frac{1}{4}\delta \cdot a \quad \text{for } x \in \left[y_N, y_N + \frac{1}{4}\delta\right]$$

and/or

$$\big|f^{(r-1)}(x)\big| > \frac{1}{4}\delta \cdot a \quad \text{for } x \in \left[y_N + \frac{3}{4}\delta, y_N + \delta\right].$$

(For example, if $f^{(r)}(x) < -a$ for $x \in [y_N, y_N + \delta]$ and $f^{(r-1)}\big(y_N + \frac{1}{2}\delta\big) \leqslant 0$ then the second inequality holds.) On the analogy of this we find that there exists an interval $\big[y, y + \frac{1}{4^{r-1}}\delta\big] \subset [y_N, y_N + \delta]$ such that $|f'(x)| > \delta^{r-1}a/4^{1+2+\cdots+(r-1)}$ for $x \in \big[y, y + \frac{1}{4^{r-1}}\delta\big]$. Therefore

$$\left|f\left(y + \frac{1}{4^{r-1}}\delta\right) - f(y)\right| > \delta^r a/4^{(r+2)(r-1)/2}.$$

Note that X is arbitrary and $y > X$, we find that the inequality contradicts the existence of $\lim_{x\to+\infty} f(x)$. This completes the proof of Lemma 1. □

Lemma 2. *Consider a C^{m+p} function $f:[0,+\infty)\to R$, if $f(x)$ satisfies*

(i) *$f^{(m)}(x)$ is uniformly continuous on* $[0,+\infty)$,
(ii) *the limit* $\lim_{x\to+\infty} f(x)$ *exists,*

then there exists a sequence $\{t_k\}$ *with* $t_k\to+\infty$ $(k\to\infty)$ *such that*

$$\lim_{k\to\infty}\left|f'(t_k)\right|+\left|f''(t_k)\right|+\cdots+\left|f^{(m)}(t_k)\right|+\left|f^{(m+p)}(t_k)\right|=0 \tag{2}$$

holds.

Proof. We show first that

$$\lim_{x\to+\infty} f^{(m)}(x)=0. \tag{3}$$

Suppose that this is not true, then there exists a sequence $\{y_k\}$ with $y_k\to+\infty$ $(k\to\infty)$ such that $\lim_{k\to\infty} f^{(m)}(y_k)=c\neq 0$. By Lemma 1, there exists a sequence $\{x_k\}$ with $x_k-y_k\to 0$ $(k\to\infty)$ such that $\lim_{k\to\infty} f^{(m)}(x_k)=0$, therefore $\lim_{k\to\infty}|f^{(m)}(x_k)-f^{(m)}(y_k)|=|c|\neq 0$. This contradicts condition (i). Since (3) holds, we know that $f^{(m)}(x)$ is bounded on $[0,+\infty)$. Therefore $f^{(m-1)}(x)$ is uniformly continuous on $[0,+\infty)$ and $\lim_{x\to+\infty} f^{(m-1)}(x)=0$. On the analogy of this, we have

$$\lim_{x\to+\infty} f^{(r)}(x)=0,\quad 1\leqslant r\leqslant m. \tag{4}$$

Moreover, using Lemma 1, we know that there exists a sequence $\{t_k\}$ with $t_k\to+\infty$ $(k\to\infty)$ such that

$$\lim_{k\to\infty} f^{(m+p)}(t_k)=0. \tag{5}$$

By (4) and (5) we know that (2) holds. This completes the proof of Lemma 2. □

3. Main results

Now we give the main results of this paper.

Theorem 1. *Consider differential equations* (1) *under assumptions above. Besides we assume that $f(t,x)$ is a C^{m+p-1} function on the set $[0,+\infty)\times B_H$. If there exists a C^{m+p} function $V(t,x):[0,+\infty)\times B_H\to R$ such that the following conditions are fulfilled on the set $[0,+\infty)\times B_H$:*

(i) *$a(\|x\|)\leqslant V(t,x)\leqslant b(\|x\|)$, where $a,b\in K$; K is the class of Hahn's functions* [1],
(ii) *$\frac{dV}{dt}\leqslant 0$, where $\frac{dV}{dt}=\frac{\partial V}{\partial t}+\left(\frac{\partial V}{\partial x}\mid f(t,x)\right)$; $(\cdot\mid\cdot)$ designates any scalar product in R^n,*

(iii) $\frac{d^{m+1}V}{dt^{m+1}}$ *is bounded on the set* $[0, +\infty) \times B_H$, *where*

$$\frac{d^{k+1}V}{dt^{k+1}} = \frac{\partial\left(\frac{d^kV}{dt^k}\right)}{\partial t} + \left(\frac{\partial\left(\frac{d^kV}{dt^k}\right)}{\partial x} \,\middle|\, f(t,x)\right), \quad k = 1, 2, \ldots, m,$$

(iv) $U(t,x) \overset{\text{def}}{=} -\left(\left|\frac{dV}{dt}\right| + \left|\frac{d^2V}{dt^2}\right| + \cdots + \left|\frac{d^mV}{dt^m}\right| + \left|\frac{d^{m+p}V}{dt^{m+p}}\right|\right) \leqslant -c(\|x\|)$, *where* $c \in K$,

then the solution

$$x = 0$$

of differential equations (1) *is asymptotically stable.*

Proof. From conditions (i) and (ii) it follows that the zero solution $x = 0$ of Eq. (1) is uniformly stable. Therefore for any $t_0 > 0$ and $h \in (0, H)$ there exists $\delta > 0$ such that any solution $x(t)$ of Eq. (1) satisfies $\|x(t)\| < h$ for every $t > t_0$, if $\|x(t_0)\| < \delta$. Choose such $\delta > 0$, we will show that any solution $x(t)$ with $\|x(t_0)\| < \delta$ satisfies

$$\lim_{t\to+\infty} x(t) = 0.$$

Denote $v(t) = V(t, x(t))$, then from condition (i) we have $v(t) \geqslant 0$, from condition (ii) we know that $v(t)$ is monotonically nonincreasing. Therefore the limit $\lim_{t\to+\infty} v(t)$ exists. From condition (iii) we know that $v^{(m+1)}(t)$ is bounded. Thus $v^{(m)}(t)$ is uniformly continuous. It follows from Lemma 2 that there exists a sequence $\{t_k\}$ with $t_k \to +\infty$ $(k \to \infty)$ such that

$$\lim_{k\to\infty} -\left(\left|v'(t_k)\right| + \left|v''(t_k)\right| + \cdots + \left|v^{(m)}(t_k)\right| + \left|v^{(m+p)}(t_k)\right|\right) = 0. \tag{6}$$

Note that

$$U\bigl(t, x(t)\bigr) = -\left(\left|v'(t)\right| + \left|v''(t)\right| + \cdots + \left|v^{(m)}(t)\right| + \left|v^{(m+p)}(t)\right|\right),$$

thus equality (6) implies

$$\lim_{k\to\infty} U\bigl(t_k, x(t_k)\bigr) = 0.$$

From this equality and condition (iv) it follows that $\lim_{k\to\infty} x(t_k) = 0$. Since the zero solution of Eq. (1) is uniformly stable we have $\lim_{t\to+\infty} x(t) = 0$. Therefore the zero solution $x = 0$ of Eq. (1) is asymptotically stable. The proof of Theorem 1 is complete. □

Now we give the corresponding instability criterion.

Theorem 2. *Let differential equations* (1) *satisfy the same assumptions as in Theorem* 1. *If there exists a* C^{m+p} *function* $V(t,x) : [0, +\infty) \times B_H \to R$, $V(t, 0) = 0$ *such that the following conditions are fulfilled*:

(i) *for any given* $t > 0$ *and* $\varepsilon > 0$, $B_\varepsilon \cap D_t$ *is a nonempty set, where* $B_\varepsilon = \{x \in R^n: \|x\| \leqslant \varepsilon\}$, $D_t = \{x \in B_H: V(t,x) > 0\}$,

(ii) $V(t,x)$ *is bounded on the set* $G = \{(t,x) \in [0, +\infty) \times B_H: V(t,x) > 0\}$,

(iii) $\frac{dV}{dt} \geqslant 0$ *on* G,

(iv) $\frac{d^{m+1}V}{dt^{m+1}}$ *is bounded on* G,

(v) *for any* $a > 0$ *there exists* $l > 0$ *such that on the set* $\Omega = \{(t, x) \in [0, +\infty) \times B_H$: $V(t, x) \geqslant a\}$, $W(t, x) \overset{\text{def}}{=} \left(\left|\frac{dV}{dt}\right| + \left|\frac{d^2V}{dt^2}\right| + \cdots + \left|\frac{d^mV}{dt^m}\right| + \left|\frac{d^{m+p}V}{dt^{m+p}}\right|\right) \geqslant l$,

then the solution $x = 0$ *of differential equations* (1) *is unstable.*

Proof. Let $h \in (0, H)$, we take arbitrary $t_0 \geqslant 0$ and arbitrary small $\delta > 0$. By condition (i) there exists x_0, $\|x_0\| < \delta$ such that $V(t_0, x_0) > 0$. Now consider the solution $x(t)$ of Eq. (1) which satisfies the initial value condition $x(t_0) = x_0$. We will show that there exists $t_1 > t_0$ such that $\|x(t_1)\| \geqslant h$. Suppose that this is not true, i.e., inequality

$$\left\|x(t_1)\right\| < h$$

holds for each $t_1 > t_0$. From condition (iii) we derive that

$$V\big(t, x(t)\big) \geqslant V\big(t_0, x(t_0)\big) = V(t_0, x_0) \overset{\text{def}}{=} a > 0 \quad \text{for } t \geqslant t_0 \tag{7}$$

and $V(t, x(t))$ is nondecreasing, by condition (ii) we know that $V(t, x(t))$ is bounded. Therefore the limit of $v(t) \overset{\text{def}}{=} V(t, x(t))$ as $t \to +\infty$ exists. By condition (iv) we know that $v^{(m)}(t)$ is uniformly continuous. By Lemma 2 there exists a sequence $\{t_k\}$ with $t_k \to +\infty$ $(k \to \infty)$ such that

$$\lim_{k\to\infty} \left(\left|v'(t_k)\right| + \left|v''(t_k)\right| + \cdots + \left|v^{(m)}(t_k)\right| + \left|v^{(m+p)}(t_k)\right|\right) = 0.$$

Since

$$W\big(t, x(t)\big) = \left|v'(t)\right| + \left|v''(t)\right| + \cdots + \left|v^{(m)}(t)\right| + \left|v^{(m+p)}(t)\right|,$$

we have

$$\lim_{k\to\infty} W\big(t_k, x(t_k)\big) = 0. \tag{8}$$

From condition (v) we know that equality (8) contradicts inequality (7). The proof of Theorem 2 is complete. □

4. Examples

Example 1. Consider the system of ordinary differential equations

$$\frac{dx}{dt} = -\sin y - h(t)x, \quad \frac{dy}{dt} = x, \tag{9}$$

where $h = h(t) = 2 + \sin\frac{\pi}{2}\sqrt{t} - \sin\frac{\pi}{2}t$. The zero solution of (9) is asymptotically stable.

Proof. Using $V = \frac{x^2}{2} + (1 - \cos y)$, we have

$$\begin{aligned}
\frac{dV}{dt} &= -hx^2 \leqslant 0,\\
\frac{d^2V}{dt^2} &= 2hx\sin y + x^2(2h^2 - h') \stackrel{\text{def}}{=} -h'x^2 + hf_1(t,x,y),\\
\frac{d^3V}{dt^3} &= (4h' - 6h^2)x\sin y - 2h\sin^2 y + (2h\cos y - 4h^3 + 6hh' - h'')x^2\\
&\stackrel{\text{def}}{=} -2h\sin^2 y + xf_2(t,x,y),\\
\frac{d^4V}{dt^4} &= \cdots \stackrel{\text{def}}{=} -6h'\sin^2 y + 6h^2\sin^2 y + xf_3(t,x,y).
\end{aligned}$$

From the boundedness of $f_1(t,x,y)$, $f_2(t,x,y)$ and $f_3(t,x,y)$ on $[1,+\infty)\times B_H$, it is not hard to derive that there exists a function $c \in K$ (K is the class of Hahn's functions) such that the following inequalities hold on the set $[1,+\infty)\times B_H$ (where H is a sufficiently small positive number):

$$\begin{aligned}
&\left|\frac{dV}{dt}\right| + \left|\frac{d^3V}{dt^3}\right| = hx^2 + \left|2h\sin^2 y - xf_2(t,x,y)\right| \geqslant c(x^2+y^2)\\
&\quad \text{for } t \notin \bigcup_{k=1}^{\infty}\left[(4k-1)^2-\delta,(4k-1)^2+\delta\right],\\
&\left|\frac{d^2V}{dt^2}\right| + \left|\frac{d^4V}{dt^4}\right| = \left|h'x^2 - hf_1(t,x,y)\right| + \left|6h'\sin^2 y - 6h^2\sin^2 y - xf_3(t,x,y)\right|\\
&\qquad \geqslant c(x^2+y^2) \quad \text{for } t \in \bigcup_{k=1}^{\infty}\left[(4k-1)^2-\delta,(4k-1)^2+\delta\right],
\end{aligned}$$

where $t = (4k-1)^2$ ($k = 1,2,\ldots,n,\ldots$) is zero point of function $h(t)$, δ is some small positive number. Thus

$$-\left(\left|\frac{dV}{dt}\right| + \left|\frac{d^2V}{dt^2}\right| + \left|\frac{d^3V}{dt^3}\right| + \left|\frac{d^4V}{dt^4}\right|\right) \leqslant -c(x^2+y^2)$$

holds on the set $[1,+\infty)\times B_H$. Moreover, the other conditions of Theorem 1 are all satisfied. Therefore, by Theorem 1, the zero solution of (9) is asymptotically stable. □

Example 2. Consider the system of ordinary differential equations

$$\frac{dx}{dt} = -\sin y + h(t)x, \quad \frac{dy}{dt} = x, \tag{10}$$

where $h(t) = 2 + \sin\frac{\pi}{2}\sqrt{t} - \sin\frac{\pi}{2}t$. By Theorem 2, using $V = \frac{x^2}{2} + (1-\cos y)$ one can show that the zero solution of the system (10) is unstable.

References

[1] N. Rouche, P. Habets, M. Laloy, Stability Theory by Liapunov's Direct Method, Springer-Verlag, New York, 1977.